

# Spinning $Q$ -balls in Abelian Gauge Theories with positive potentials: existence and non existence

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## Abstract

We study the existence of cylindrically symmetric electro-magneto-static solitary waves for a system of a nonlinear Klein–Gordon equation coupled with Maxwell’s equations in presence of a positive mass and of a nonnegative nonlinear potential. Nonexistence results are provided as well.

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## 1 Introduction, motivations and results

In recent past years great attention has been paid to some classes of systems of partial differential equations that provide a model for the interaction of matter with the electromagnetic field. Such theories are known in literature as Abelian Gauge Theories, and in this framework a crucial rôle is played by systems whose field equation is the Klein–Gordon one. In particular, we recall the papers [1], [2], [3], [4], [5], [6], [7], [11], [12], [15], [18], [22], [23], [24], [26] and [30], where existence or non existence results are proved in the whole physical space for systems of the Klein-Gordon-Maxwell type.

Here we are interested in a particular class of solutions, consisting in the so called *solitary waves*, i.e. solutions of a field equation whose energy travels as a localized packet. This kind of solutions plays an important rôle in these theories because of their relationship with *solitons*. “Soliton” is the name by which solitary waves are known when they exhibit some strong form of stability; they appear in many situations of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics (for example see [13], [17] and [27]). Therefore, the first step to prove the existence of solitons is to prove the existence of solitary waves, as we will do.

Our starting point is the following system, obtained by the interaction of a Klein–Gordon field with Maxwell’s equations, which is, therefore, a model for electrodynamics (for the derivation of the general system and for a detailed description of the physical meaning of the unknowns we refer to the papers cited above and their references):

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + |\nabla S - q\mathbf{A}|^2 u - \left(\frac{\partial S}{\partial t} + q\phi\right)^2 u + W'(u) = 0, \\ \frac{\partial}{\partial t} \left[ \left(\frac{\partial S}{\partial t} + q\phi\right) u^2 \right] - \operatorname{div}[(\nabla S - q\mathbf{A})u^2] = 0, \\ \operatorname{div}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi\right) = q\left(\frac{\partial S}{\partial t} + q\phi\right) u^2, \\ \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi\right) = q(\nabla S - q\mathbf{A})u^2, \end{cases} \quad (1.1)$$

where the equations are the matter equation, the charge continuity equation, the Gauss equation and the Ampère equation, respectively.

We are interested in standing waves, i.e. solutions having the special form

$$\psi(t, x) = u(x)e^{iS(x,t)}, \quad u : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad S(x, t) = S_0(x) - \omega t \in \mathbb{R}, \quad \omega \in \mathbb{R}, \quad (1.2)$$

$$\partial_t \mathbf{A} = 0, \quad \partial_t \phi = 0. \quad (1.3)$$

Three different types of finite energy, stationary nontrivial solutions can be considered:

- electrostatic solutions:  $\mathbf{A} = 0, \phi \neq 0$ ;
- magnetostatic solutions:  $\mathbf{A} \neq 0, \phi = 0$ ;
- electro-magneto-static solutions:  $\mathbf{A} \neq 0, \phi \neq 0$ .

Under suitable assumptions, all these types of solutions may exist.

Existence and nonexistence of electrostatic solutions for system (1.1) have been proved under different assumptions on  $W$ : in [11] and [12] the following, or more general, potential has been taken into account:

$$W(s) = \frac{1}{2}s^2 - \frac{s^p}{p}, \quad s \geq 0.$$

In particular, in [4] the case  $4 < p < 6$ , in [12] the case  $2 < p < 6$  and in [11] the remaining cases are considered.

In [2] and [26] the existence of electrostatic solutions has been studied for the first time when the potential  $W$  is *positive*. In particular the existence of radially symmetric, electrostatic solutions has been analyzed in both cases, and it turns out that all these solutions have zero angular momentum.

Here we are interested in electro-magneto-static solutions when  $W \geq 0$ ; in particular, we shall study the existence of vortices, which are solutions with non vanishing angular momentum, namely solutions with  $S_0(x) = l\theta(x) - \theta$  being a suitable function we will introduce later -, i.e. of the form

$$\psi(t, x) = u(x)e^{i(l\theta(x) - \omega t)}, \quad l \in \mathbb{Z} \setminus \{0\}, \quad (1.4)$$

and we will see that the angular momentum  $\mathbf{M}_m$  of the matter field of a vortex does not vanish (see Remark 2.3); this fact justifies the name “vortex”.

These kind of solutions are also known as *spinning  $Q$ -balls*; in this regard we recall the pioneering paper of Rosen [29] and of Coleman [10]. Coleman was the first one to use the name  *$Q$ -ball* in that paper, referring to spherically symmetric solutions. Vortices in the nonlinear Klein–Gordon–Maxwell equations (with a positive nonlinear term  $W(s)$  with  $W(0) = 0$ ) are also considered in Physics literature with the name of *spinning  $Q$ -balls*, even if they do not exhibit a spherical symmetry, as in the case treated in this paper.

Our attention is concentrated on  $W$  and in particular on the fact that it is assumed *nonnegative* and it possesses some good invariants (necessary to be considered in Abelian Gauge Theories), typically some conditions of the form

$$W(e^{i\alpha}u) = W(u) \quad \text{and} \quad W'(e^{i\alpha}u) = e^{i\alpha}W'(u)$$

for any function  $u$  and any  $\alpha \in \mathbb{R}$ . Thanks to these assumptions, the system becomes

$$-\Delta u + [l\nabla\theta - q\mathbf{A}]^2 - (\omega - q\phi)^2 u + W'(u) = 0, \quad (1.5)$$

$$-\Delta\phi = q(\omega - q\phi)u^2, \quad (1.6)$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(l\nabla\theta - q\mathbf{A})u^2, \quad (1.7)$$

which is the Klein–Gordon–Maxwell system we have investigated. Moreover, though system (1.5)–(1.6)–(1.7) was obtained by means of considerations on gauge invariance of  $W$ , from a mathematical point of view we can also replace (1.5) with

$$-\Delta u + [l\nabla\theta - q\mathbf{A}]^2 - (\omega - q\phi)^2 u + W_u(x, u) = 0,$$

i.e. we could let  $W$  depend on the  $x$ -variable. More precisely, in order to use our functional approach, we let  $W$  depend on  $(\sqrt{x_1^2 + x_2^2}, x_3)$ , but we do not require any positivity far from 0, in contrast to the usual Ambrosetti–Rabinowitz condition. We think that this fact is very interesting, both from a mathematical and a physical point of view: for example, it may happen that the potential is inactive in some cylinder, or, even more interesting, out of a cylinder, as it happens where strong magnetic potential are present in linear accelerators.

According to what just said, in the second section we will show a new existence result for system (1.5)–(1.6)–(1.7) under general assumptions on the nonnegative potential  $W$ . We were inspired by the approach of [5], and for this reason, the functional structure is the same one of that article. However, our hypotheses on  $W$  imply, in particular, that the potential  $W(s)$  might be 0 for values of  $s$  different from 0, in contrast to [5] where the potential  $W$  was assumed to lie above a parabola. This corresponds to the situation in which, for values of the unknown different from 0, there is no interaction among particles (see [12], [26]). Moreover, even more interesting, the existence result given in [5] is valid only for small values of the charge  $q$ , while we remove such an assumption.

In conclusion, though our assumptions are weaker, our results are stronger than those in [5].

Entering into details, we shall study system (1.5)–(1.6)–(1.7) under the following hypothesis on the potential  $W$ :

W1)  $W(s) \geq 0$  for all  $s \geq 0$ ;

W2)  $W$  is of class  $C^2$  with  $W(0) = W'(0) = 0, W''(0) = m^2 > 0$ ;

W3) setting

$$W(s) = \frac{m^2}{2}s^2 + N(s), \quad (1.8)$$

we assume that there exist positive constants  $c_1, c_2, p, \ell$ , with  $2 < \ell \leq p < 6$ , such that for all  $s \geq 0$  there holds

$$|N'(s)| \leq c_1 s^{\ell-1} + c_2 s^{p-1}.$$

Moreover, though we are interested in positive solutions, it is convenient to extend  $W$  for all  $s \in \mathbb{R}$  setting

$$W(s) = W(-s) \text{ for every } s < 0.$$

System (1.5)–(1.6)–(1.7) was introduced in [5] assuming W1), W2), W3) and their fundamental requirement

$$\inf_{s>0} \left( \frac{W(s)}{\frac{m^2}{2}s^2} \right) < 1. \quad (1.9)$$

We immediately see that assumption W3) plus (1.9) is equivalent to require that there exists  $s_0 > 0$  such that  $N(s_0) < 0$ , the first step in the classical “Berestycki–Lions” approach. In this paper we will use an hypothesis different from (1.9), which will let us prove our main result without any restriction on the charge  $q$ , as in [5]. Indeed, we will assume

W4) there exist  $\varepsilon_0 > 0, D > 0$  and  $\tau > 2$  such that

$$N(s) \leq -D|s|^\tau \quad \text{for all } s \in [0, \varepsilon_0].$$

It is clear that functions of the type  $N(s) = |s|^p - |s|^q$ ,  $2 < q < p$ , satisfy W4).

**Remark 1.1.** If we consider the electrostatic case, i.e.  $-\Delta u + W'(u) = 0$ , calling “rest mass” of the particle  $u$  the quantity

$$\int_{\mathbb{R}^3} W(u) dx,$$

see [7], our assumptions on  $W$  imply that we are dealing *a priori* with systems for particles having *positive mass*, which is, of course, the physical interesting case.

As usual, for physical reasons, we look for solutions having finite energy, i.e.  $(u, \phi, \mathbf{A}) \in H^1 \times \mathcal{D}^1 \times (\mathcal{D}^1)^3$ , where  $H^1 = H^1(\mathbb{R}^3)$  is the usual Sobolev space, and  $\mathcal{D}^1 = \mathcal{D}^1(\mathbb{R}^3)$  is the completion of  $\mathcal{D} = C_c^\infty(\mathbb{R}^3)$  with respect to the norm  $\|u\|_{\mathcal{D}^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 dx$  (see Section 2.2 for the precise functional setting).

Before giving our main result, we remark that, as in [5], the parameter  $\omega$  is an unknown of the problem.

**Theorem 1.2.** *Assume W1), W2), W3) and W4). For every  $l \in \mathbb{Z}$  and  $q \geq 0$  there exists  $D_0 > 0$  such that, if W4) holds with  $D \geq D_0$ , then system (1.5)–(1.6)–(1.7) admits a finite energy solution in the sense of distributions  $(u, \omega, \phi, \mathbf{A})$ ,  $u \neq 0, \omega > 0$  such that*

- the maps  $u, \phi$  depend only on the variables  $r = \sqrt{x_1^2 + x_2^2}$  and  $x_3$ ;

•

$$\int_{\mathbb{R}^3} \frac{u^2}{r^2} dx \in \mathbb{R};$$

- the magnetic potential  $\mathbf{A}$  has the following form:

$$\mathbf{A} = a(r, x_3) \nabla \theta = a(r, x_3) \left( \frac{x_2}{r^2} \mathbf{e}_1 - \frac{x_1}{r^2} \mathbf{e}_2 \right). \quad (1.10)$$

If  $q = 0$ , then  $\phi = 0$ ,  $\mathbf{A} = \mathbf{0}$ . If  $q > 0$ , then  $\phi \neq 0$ . Moreover,  $\mathbf{A} \neq \mathbf{0}$  if and only if  $l \neq 0$ .

**Remark 1.3.** As it will be clear from the proof,  $D_0$  is independent of  $q$ , but depends only on  $m, l$  and  $\sigma$ .

**Remark 1.4.** Noether’s Theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion. For our purposes, the most relevant integral is the angular momentum. By definition, the angular momentum is the quantity which is preserved by virtue of the invariance under space rotations of the Lagrangian with respect to the origin. Using the gauge invariant variables, we get:

$$\mathbf{M} = \mathbf{M}_m + \mathbf{M}_f,$$

where

$$\mathbf{M}_m = \int_{\mathbb{R}^3} \left[ -x \times (\nabla u \partial_t u) + x \times \frac{\rho \mathbf{j}}{q^2 u^2} \right] dx$$

and

$$\mathbf{M}_f = \int_{\mathbb{R}^3} x \times (\mathbf{E} \times \mathbf{H}) dx.$$

Here  $\mathbf{M}_m$  refers to the “matter field” and  $\mathbf{M}_f$  to the “electromagnetic field”, while  $\rho$  and  $\mathbf{j}$  denote the electric charge and the current density, respectively.

We will see below that the solution found in Theorem 1.2 have nontrivial angular momentum, see Remark 2.3.

**Remark 1.5.** When  $l = 0$  and  $q > 0$  the last part of Theorem 1.2 states the existence of electrostatic solutions, namely finite energy solutions with  $u \neq 0$ ,  $\phi \neq 0$  and  $\mathbf{A} = \mathbf{0}$ . This result is a variant of a recent ones (see [2] and [26]).

Moreover, let us observe that under general assumptions on  $W$ , magneto-static solutions (i.e. with  $\omega = \phi = 0$ ) do not exist. In fact also the following proposition is proved in [5]:

**Remark 1.6** ([5], Prop.8 p.649). Assume that  $W$  satisfies the assumptions  $W(0) = 0$  and  $W'(s)s \geq 0$ . Then (1.5), (1.6), (1.7) has no solutions with  $\omega = \phi = 0$ .

In our setting, we are able to prove the following nonexistence results:

**Theorem 1.7.** *If  $u$  is a finite energy solution of (1.5) with*

$$\int_{\mathbb{R}^3} N(u) dx \in \mathbb{R},$$

and

- $\omega^2 < m^2$  and either  $N \geq 0$  or  $N'(s)s \leq 6N(s)$  for all  $s \in \mathbb{R}$ ,
- or
- $N'(s)s \geq 2N(s)$  for all  $s \in \mathbb{R}$ ,

then  $u \equiv 0$ .

A natural consequence is the following

**Corollary 1.8.** *If  $u \in L^p(\mathbb{R}^3)$  is a finite energy solution of (1.5)–(1.6)–(1.7), and*

- $\omega^2 < m^2$  and

$$N(u) = \begin{cases} \frac{|u|^p}{p}, & p \leq 6, \\ -\frac{|u|^p}{p}, & p \geq 6, \end{cases}$$

or

•

$$N(u) = \begin{cases} \frac{|u|^p}{p}, & p \geq 2, \\ -\frac{|u|^p}{p}, & p \leq 2, \end{cases}$$

then  $u \equiv 0$ .

**Remark 1.9.** Theorem 1.7 implies that, in general, in order to have vortices with  $N \geq 0$  it is necessary to have a “large” frequency. We are not aware of similar results in the theory of vortices, and we believe such a result can shed a new light on this subject.

In Section 4 we shall prove another existence result concerning a different kind of solutions, namely solutions having fixed  $L^2$  norm. In general these solutions cannot be obtained from the solutions found in Theorem 1.2, for example via a rescaling argument, and we shall focus on the case  $\int_{\mathbb{R}^3} u^2 dx = 1$ , which corresponds to look for solutions having a density of probability equal to 1. An analogous result could be obtained for  $\int_{\mathbb{R}^3} u^2 dx = c \in \mathbb{R}^+$ , but the physical meaning of this kind of solutions is not clear to us. Indeed, in different situations it may happen that if  $\int_{\mathbb{R}^3} u^2 = c$  is fixed a priori, then solutions appear only for certain values of  $c$ : a typical example is in the context of boson stars, when solutions with fixed energy do exist if and only if  $c < M_C$ , the Chandrasekhar limit mass (see [19] and [25]).

Our result is the following

**Proposition 1.10.** *Under the hypotheses of Theorem 1.2, there exists  $\mu \in \mathbb{R}$  and a solution in the sense of distributions for the system*

$$\begin{aligned} -\Delta u + [l\nabla\theta - q\mathbf{A}]^2 u + W'(u) &= \mu u, \\ -\Delta\phi &= q(\omega - q\phi)u^2, \\ \nabla \times (\nabla \times \mathbf{A}) &= q(l\nabla\theta - q\mathbf{A})u^2, \end{aligned}$$

such that  $\int_{\mathbb{R}^3} u^2 dx = 1$ . Moreover, if  $\omega^2 \leq m^2$  and  $N'(s)s \geq 0$  for all  $s \in \mathbb{R}$ , then  $\mu > 0$ .

Due to the presence of the multiplier  $\mu$ , we give the following

**Definition 1.11.** We call *effective mass* of the system the quantity  $\tilde{m} = m^2 - \mu$ .

## 2 Preliminary setting

### 2.1 Standing wave solutions and vortices

Substituting (1.2) and (1.3) in (1.1), we get the following equations in  $\mathbb{R}^3$ :

$$-\Delta u + \left[ |\nabla S_0 - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] u + W'(u) = 0, \quad (2.11)$$

$$-\operatorname{div} \left[ (\nabla S_0 - q\mathbf{A}) u^2 \right] = 0, \quad (2.12)$$

$$-\Delta \phi = q(\omega - q\phi) u^2, \quad (2.13)$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(\nabla S_0 - q\mathbf{A}) u^2. \quad (2.14)$$

We can easily observe that (2.12) follows from (2.14): as a matter of fact, applying the divergence operator to both sides of (2.14), we immediately get (2.12). Then we are reduced to study the system (2.11)–(2.13)–(2.14).

We are interested in finite-energy solutions - the most relevant physical case - i.e. solutions of system (2.11)–(2.13)–(2.14) for which the following energy is finite:

$$\begin{aligned} \mathcal{E} = & \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 + (|\nabla S_0 - q\mathbf{A}|^2 + (\omega - q\phi)^2) u^2 \right) dx \\ & + \int_{\mathbb{R}^3} W(u) dx \end{aligned} \quad (2.15)$$

Moreover the (electric) charge (see e.g. [5], p.644) is given by

$$Q = q\sigma, \quad (2.16)$$

where

$$\sigma = \int_{\mathbb{R}^3} (\omega - q\phi) u^2 dx. \quad (2.17)$$

**Remark 2.1.** When  $u = 0$ , the only finite energy gauge potentials which solve (2.13), (2.14) are the trivial ones  $\mathbf{A} = \mathbf{0}, \phi = 0$ .

In particular, following [5], we shall look for solutions of the system above which are known in literature as vortices. In order to do that, we need some preliminaries. First, set

$$\Sigma = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0 \right\},$$

and define the map

$$\begin{aligned} \theta : \mathbb{R}^3 \setminus \Sigma &\rightarrow \frac{\mathbb{R}}{2\pi\mathbb{Z}}, \\ \theta(x_1, x_2, x_3) &= \operatorname{Im} \log(x_1 + ix_2). \end{aligned}$$

The following definition is crucial:



**Definition 2.2.** A finite energy solution  $(u, S_0, \phi, \mathbf{A})$  of (2.11)–(2.13)–(2.14) is called *vortex* if  $S_0 = l\theta$  for some  $l \in \mathbb{Z} \setminus \{0\}$ .

Of course, in this case,  $\psi$  has the form

$$\psi(t, x) = u(x)e^{i(l\theta(x) - \omega t)}, \quad l \in \mathbb{Z} \setminus \{0\}. \quad (2.18)$$

**Remark 2.3.** In [5, Proposition 7] it was proved that if  $(u, \omega, \phi, \mathbf{A})$  is a non trivial, finite energy solution of (2.11)–(2.13)–(2.14), then the angular momentum  $\mathbf{M}_m$  has the expression

$$\mathbf{M}_m = - \left[ \int_{\mathbb{R}^3} (l - qa)(\omega - q\phi)u^2 dx \right] \mathbf{e}_3, \quad (2.19)$$

and, if  $l \neq 0$ , it does not vanish. Hence, in this case, the name “vortex” is justified and by Theorem 1.2 the existence of a spinning  $Q$ -ball is guaranteed.

Now, observe that  $\theta \in C^\infty(\mathbb{R}^3 \setminus \Sigma, \frac{\mathbb{R}}{2\pi\mathbb{Z}})$ , and, with abuse of notation, we set

$$\nabla\theta(x) = \frac{x_2}{x_1^2 + x_2^2} \mathbf{e}_1 - \frac{x_1}{x_1^2 + x_2^2} \mathbf{e}_2,$$

where  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  is the standard frame in  $\mathbb{R}^3$ .

Using the *Ansatz* (2.18), equations (2.11), (2.13), (2.14) give raise to equations (1.5), (1.6), (1.7), which is the Klein–Gordon–Maxwell system we shall study from now on.

**Remark 2.4.** If  $\mathbf{A} = \left( \frac{x_2}{x_1^2 + x_2^2}, -\frac{x_1}{x_1^2 + x_2^2}, 0 \right)$ , we obviously get  $\nabla \times \mathbf{A} = 0$ .

*Viceversa*, if  $\mathbf{A}$  is irrotational and it solves (1.7), then  $\mathbf{A} = \frac{l}{q} \nabla\theta$ . In such a case, system (1.5)–(1.6)–(1.7) reduces to the one considered in [26], where, by Theorem 1.7, we can now say that the nontrivial solution found therein is such that  $\omega^2 \geq m^2$ .

## 2.2 Functional approach

We shall follow the functional approach of [5], with minor changes in some parts. Anyway, our main Theorem 1.2 has been proved thanks to completely new results (see Lemma 3.4 and Proposition 3.5), which let us avoid any bound on  $q$ , as in [5].

First, we denote by  $L^p \equiv L^p(\mathbb{R}^3)$  ( $1 \leq p < +\infty$ ) the usual Lebesgue space endowed with the norm

$$\|u\|_p^p := \int_{\mathbb{R}^3} |u|^p dx.$$

We also recall the continuous embeddings

$$H^1(\mathbb{R}^3) \hookrightarrow D^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \quad \text{and} \quad H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3) \quad \forall p \in [2, 6], \quad (2.20)$$

being 6 the critical exponent for the Sobolev embedding  $\mathcal{D}^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ . Here  $H^1 \equiv H^1(\mathbb{R}^3)$  denote the usual Sobolev space with norm

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx$$

and  $\mathcal{D}^1 = \mathcal{D}^1(\mathbb{R}^3)$  is the completion of  $\mathcal{D} = C_c^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{\mathcal{D}^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 dx,$$

induced by the scalar product  $(u, v)_{\mathcal{D}^1} := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx$ .

Moreover, we need the weighted Sobolev space  $\hat{H}^1 \equiv \hat{H}_l^1(\mathbb{R}^3)$ , depending on a fixed integer  $l$ , whose norm is given by

$$\|u\|_{\hat{H}^1}^2 = \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + \left(1 + \frac{l^2}{r^2}\right) u^2 \right] dx, \quad l \in \mathbb{Z},$$

where  $r = \sqrt{x_1^2 + x_2^2}$ . Clearly  $\hat{H}^1 = H^1$  if and only if  $l = 0$ . Moreover, it is not hard to see that

$$C_c^\infty(\mathbb{R}^3) \cap \hat{H}^1(\mathbb{R}^3) \text{ is dense in } \hat{H}^1(\mathbb{R}^3). \quad (2.21)$$

We set

$$\begin{aligned} H &= \hat{H}^1 \times \mathcal{D}^1 \times (\mathcal{D}^1)^3, \\ \|(u, \phi, \mathbf{A})\|_H^2 &= \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + \left(1 + \frac{l^2}{r^2}\right) u^2 + |\nabla \phi|^2 + |\nabla \mathbf{A}|^2 \right] dx. \end{aligned}$$

We shall denote by  $u = u(r, x_3)$  any real function in  $\mathbb{R}^3$  which depends only on the cylindrical coordinates  $(r, x_3)$ , and we set

$$\mathcal{D}_\# = \left\{ u \in \mathcal{D} : u = u(r, x_3) \right\}.$$

Finally, we shall denote by  $\mathcal{D}_\#^1$  the closure of  $\mathcal{D}_\#$  in the  $\mathcal{D}^1$  norm and by  $\hat{H}_\#^1$  the closed subspace of  $\hat{H}^1$  whose functions are of the form  $u = u(r, x_3)$ .

Now, we consider the functional

$$\begin{aligned} J(u, \phi, \mathbf{A}) &= \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 - |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2] dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} [l \nabla \theta - q \mathbf{A}]^2 - (\omega - q \phi)^2 u^2 dx + \int_{\mathbb{R}^3} W(u) dx, \end{aligned} \quad (2.22)$$

where  $(u, \phi, \mathbf{A}) \in H$ . Formally, equations (1.5), (1.6) and (1.7) are the Euler–Lagrange equations of the functional  $J$ , and, indeed, standard computations show that the following lemma holds:

**Lemma 2.5.** *Assume that  $W$  satisfies W3). Then the functional  $J$  is of class  $C^1$  on  $H$  and equations (1.5), (1.6) and (1.7) are its Euler–Lagrange equations.*

By the above lemma it follows that any critical point  $(u, \phi, \mathbf{A}) \in H$  of  $J$  is a weak solutions of system (1.5)–(1.6)–(1.7), namely

$$\int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + [l\nabla\theta - q\mathbf{A}]^2 - (\omega - q\phi)^2] uv + W'(u)v] dx = 0 \quad \forall v \in \hat{H}^1, \quad (2.23)$$

$$\int_{\mathbb{R}^3} [\nabla\phi \cdot \nabla w - qu^2(\omega - q\phi)w] dx = 0 \quad \forall w \in \mathcal{D}^1, \quad (2.24)$$

$$\int_{\mathbb{R}^3} [(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{V}) - qu^2(l\nabla\theta - q\mathbf{A}) \cdot \mathbf{V}] dx = 0 \quad \forall \mathbf{V} \in (\mathcal{D}^1)^3. \quad (2.25)$$

### 2.3 Solutions in the sense of distributions

Since  $\mathcal{D}$  is *not* contained in  $\hat{H}^1$ , a solution  $(u, \phi, \mathbf{A}) \in H$  of (2.23), (2.24), (2.25) need not be a solution of (1.5), (1.6), (1.7) in the sense of distributions on  $\mathbb{R}^3$ . However, we will show that the singularity of  $\nabla\theta(x)$  on  $\Sigma$  is removable in the following sense:

**Theorem 2.6.** *Let  $(u_0, \phi_0, \mathbf{A}_0) \in H, u_0 \geq 0$  be a solution of (2.23), (2.24), (2.25) (i.e. a critical point of  $J$ ). Then  $(u_0, \phi_0, \mathbf{A}_0)$  is a solution of system (1.5)–(1.6)–(1.7) in the sense of distributions, namely*

$$\int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla v + [l\nabla\theta - q\mathbf{A}_0]^2 - (\omega - q\phi_0)^2] u_0 v + W'(u_0)v] dx = 0 \quad \forall v \in \mathcal{D}, \quad (2.26)$$

$$\int_{\mathbb{R}^3} [\nabla\phi_0 \cdot \nabla w - qu_0^2(\omega - q\phi_0)w] dx = 0 \quad \forall w \in \mathcal{D}, \quad (2.27)$$

$$\int_{\mathbb{R}^3} [(\nabla \times \mathbf{A}_0) \cdot (\nabla \times \mathbf{V}) - qu_0^2(l\nabla\theta - q\mathbf{A}_0) \cdot \mathbf{V}] dx = 0 \quad \forall \mathbf{V} \in (\mathcal{D})^3. \quad (2.28)$$

A proof of Theorem 2.6 was given in [5]; however, we give a similar, but different proof, in order to make precise some parts. In order to do that, we introduce a sequence of smooth functions  $\{\chi_n\}_{n \in \mathbb{N}}$  depending only on  $(r, x_3)$  and which satisfy the following assumptions:

- $\chi_n(r, x_3) = 1$  for  $r \geq \frac{2}{n}$ ,
- $\chi_n(r, x_3) = 0$  for  $r \leq \frac{1}{n}$
- $|\chi_n(r, x_3)| \leq 1$ ,
- $|\nabla\chi_n(r, x_3)| \leq 2n$ ,
- $\chi_{n+1}(r, x_3) \geq \chi_n(r, x_3)$ .

**Lemma 2.7** ([5], Lemma 11). *Let  $\varphi$  be a function in  $H^1 \cap L^\infty$  with compact support and set  $\varphi_n = \varphi \cdot \chi_n$ . Then, up to a subsequence, we have that*

$$\varphi_n \rightharpoonup \varphi \text{ in } H^1.$$

Now we are ready to give the

*Proof of Theorem 2.6.* Clearly, (2.27) and (2.28) immediately follow by (2.24) and (2.25),  $\mathcal{D}$  being contained in  $\mathcal{D}$ , so we only need to prove (2.26). The case  $l = 0$  is trivial because we do not have singularity problems and  $\mathcal{D} \subset \hat{H}^1$ . Therefore, assume  $l \neq 0$ . We take any  $v \in \mathcal{D}$  and set  $\varphi_n = v^+ \cdot \chi_n$ , where  $v^+ = \max\{v, 0\}$ . Note that  $\varphi_n \in \hat{H}^1$ , and so it can be taken as a test function in (2.23), obtaining

$$\int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla \varphi_n + (|q\mathbf{A}_0 - l\nabla\theta|^2 - (q\phi_0 - \omega)^2) u_0 \varphi_n + W'(u_0) \varphi_n] dx = 0. \quad (2.29)$$

Equation (2.29) can be written as follows

$$A_n + B_n + C_n + D_n + E_n = 0, \quad (2.30)$$

where

$$A_n = \int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla \varphi_n - \omega^2 u_0 \varphi_n + m^2 u_0 \varphi_n + 2q\omega\phi_0 u_0 \varphi_n] dx, \quad (2.31)$$

$$B_n = \int_{\mathbb{R}^3} (q^2 |\mathbf{A}_0|^2 u_0 - q^2 \phi_0^2 u_0) \varphi_n dx, \quad (2.32)$$

$$C_n = -2 \int_{\mathbb{R}^3} ql \mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n dx, \quad D_n = \int_{\mathbb{R}^3} N'(u_0) \varphi_n dx, \quad (2.33)$$

$$E_n = \int_{\mathbb{R}^3} l^2 |\nabla \theta|^2 u_0 \varphi_n dx. \quad (2.34)$$

By Lemma 2.7,

$$\varphi_n \rightharpoonup v^+ \text{ in } H^1. \quad (2.35)$$

Since  $(u_0, \phi_0, \mathbf{A}_0) \in H$ ,  $\hat{H}^1 \hookrightarrow H^1$ ,  $\mathcal{D}^1 \hookrightarrow L^6(\mathbb{R}^3)$  (by Sobolev Embedding Theorem), we have that

$$A_n \rightarrow \int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla v^+ - \omega^2 u_0 v^+ + m^2 u_0 v^+ + 2q\omega\phi_0 u_0 v^+] dx, \quad (2.36)$$

since  $\phi_0 u_0 \in L^2(\mathbb{R}^3)$ , for  $\phi_0^2 \in L^3(\mathbb{R}^3)$  and  $u_0^2 \in L^{3/2}(\mathbb{R}^3)$ .

Moreover,

$$(q^2 |\mathbf{A}_0|^2 u_0 - q^2 \phi_0^2 u_0) \in L^{6/5}(\mathbb{R}^3) = (L^6(\mathbb{R}^3))'.$$

Indeed,  $|\mathbf{A}_0|^{12/5} \in L^{5/2}$  and  $u_0^{6/5} \in L^{5/3}$ , so that  $|\mathbf{A}_0|^2 u_0 \in L^{6/5}$ ; analogously for the term  $\phi_0^2 u_0$  and the claim follows. Then, using again (2.35), we have

$$B_n \rightarrow \int_{\mathbb{R}^3} (q^2 |\mathbf{A}_0|^2 u_0 - q^2 \phi_0^2 u_0) v^+ dx < \infty. \quad (2.37)$$

Now we shall prove that

$$C_n \rightarrow -2 \int_{\mathbb{R}^3} ql \mathbf{A}_0 \cdot \nabla \theta u_0 v^+ dx < \infty. \quad (2.38)$$

For this purpose, consider the cylinder

$$C = B_R \times [-d, d], \quad B_R = \{(x_1, x_2) \in \mathbb{R}^2 : r^2 = x_1^2 + x_2^2 < R\},$$

where  $d, R > 0$  are so large that the cylinder  $C$  contains the support of  $v^+$ . Then

$$\begin{aligned} \int_{\mathbb{R}^3} \left( \frac{\varphi_n}{r} \right)^{\frac{3}{2}} dx &= \int_C \left( \frac{v^+ \chi_n}{r} \right)^{\frac{3}{2}} dx \\ &\leq c_1 \int_{-d}^d \int_0^R \left( \frac{1}{r} \right)^{\frac{3}{2}} r dr dx_3 = M < \infty, \end{aligned} \quad (2.39)$$

where  $c_1 = 2\pi(\max v^+)^{\frac{3}{2}}$ . By (2.39), by definition of  $\nabla \theta$ , which implies  $|\nabla \theta| \leq 2$ , and by interpolation, we have

$$\int_{\mathbb{R}^3} |\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| dx \leq 2 \|u_0 \mathbf{A}_0\|_{L^3} \left\| \frac{\varphi_n}{r} \right\|_{L^{\frac{3}{2}}} \leq 2 \|u_0 \mathbf{A}_0\|_{L^3} M^{\frac{2}{3}}. \quad (2.40)$$

Now

$$|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| \rightarrow |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| \text{ a.e. in } \mathbb{R}^3$$

and the sequence  $\{|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n|\}$  is monotone thanks to the monotonicity of  $\varphi_n$ . Then, by the monotone convergence theorem, we get

$$\int_{\mathbb{R}^3} |\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| dx \rightarrow \int_{\mathbb{R}^3} |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| dx. \quad (2.41)$$

By (2.40) and (2.41) we deduce that

$$\int |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| dx < \infty.$$

Moreover, we have

$$|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| \leq |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| \in L^1,$$

so, by the dominated convergence theorem, we get (2.38).

Now, by  $W3$ ) we know that  $|N'(u_0)| \leq c_1 |u_0|^{\ell-1} + c_2 |u_0|^{p-1} \in L^{6/5}$ , if  $\frac{8}{3} \leq \ell \leq p < 6$ . If at least one of the exponents, say  $\ell$ , is such that  $2 < \ell < 8/3$ , then  $|u_0|^{\ell-1} \in L^2$  by interpolation. In any case, by (2.35) we get

$$D_n \rightarrow \int_{\mathbb{R}^3} N'(u_0) v^+ dx. \quad (2.42)$$

Finally, we prove that

$$E_n \rightarrow \int_{\mathbb{R}^3} l^2 |\nabla \theta|^2 u_0 v^+ dx < \infty. \quad (2.43)$$

By (2.30), (2.36), (2.37), (2.38) and (2.42) we have that

$$E_n = \int_{\mathbb{R}^3} l^2 |\nabla \theta|^2 u_0 \varphi_n dx \text{ is bounded.} \quad (2.44)$$

Then the sequence  $\{|\nabla \theta|^2 u_0 \varphi_n\}$  is monotone and it converges a.e. to  $|\nabla \theta|^2 u_0 v^+$ . Thus, by the monotone convergence theorem, we get

$$\int_{\mathbb{R}^3} |\nabla \theta|^2 u_0 \varphi_n dx \rightarrow \int_{\mathbb{R}^3} |\nabla \theta|^2 u_0 v^+ dx. \quad (2.45)$$

By (2.44) and (2.45) we get (2.43).

Taking the limit in (2.30) and by using (2.36), (2.37), (2.38), (2.42) and (2.43), we have

$$\int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla v^+ + [q\mathbf{A}_0 - l|\nabla \theta|^2 - (q\phi_0 - \omega)^2] u_0 v^+ + W'(u_0) v^+] dx = 0.$$

Taking  $\varphi_n = v^- \cdot \chi_n$ , where  $v^- = \max\{0, -v\}$ , and arguing in the same way as before, we get

$$\int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla v^- + [q\mathbf{A}_0 - l|\nabla \theta|^2 - (q\phi_0 - \omega)^2] u_0 v^- + W'(u_0) v^-] dx = 0.$$

Then

$$\int_{\mathbb{R}^3} [\nabla u_0 \cdot \nabla v + [q\mathbf{A}_0 - l|\nabla \theta|^2 - (q\phi_0 - \omega)^2] u_0 v + W'(u_0) v] dx = 0.$$

Since  $v \in \mathcal{D}$  is arbitrary, we have proved (2.26).  $\square$

Let us now remark that the presence of the term  $-\int_{\mathbb{R}^3} |\nabla \phi|^2 dx$  gives the functional  $J$  a strong indefiniteness, namely any nontrivial critical point of  $J$  has infinite Morse index. It turns out that a direct approach to finding critical points for  $J$  is very hard. For this reason, as usual in this setting, it is convenient to introduce a *reduced functional*.

## 2.4 The reduced functional

Write equation (1.6) like

$$-\Delta \phi + q^2 u^2 \phi = q\omega u^2, \quad (2.46)$$

then we can verify that the following holds:

**Proposition 2.8** ([12], Proposition 2.2). *For every  $u \in H^1(\mathbb{R}^3)$ , there exists a unique  $\phi = \phi_u \in \mathcal{D}^1$  which solves (2.46) and there exists  $S > 0$  such that*

$$\|\phi_u\| \leq qS \|u\|_{12/5}^2 \text{ for every } u \in H^1(\mathbb{R}^3). \quad (2.47)$$

**Lemma 2.9.** *If  $u \in \hat{H}_\#^1(\mathbb{R}^3)$ , then the solution  $\phi = \phi_u$  of (2.46) belongs to  $\mathcal{D}_\#^1(\mathbb{R}^3)$ .*

*Proof.* By abuse of notation, we denote by  $O(2)$  the group of isometries in  $\mathbb{R}^3$  which act as rotations in the first two components, and we consider the  $O(2)$  group action  $T_g$  on  $H^1$  defined by

$$(T_g u)(x) = u(gx) \quad \text{for any } u \in H^1.$$

Then, if  $g \in O(2)$ , we have

$$T_g(-\Delta\phi_u + q^2 u^2 \phi_u) = q\omega T_g(u^2),$$

that is

$$-\Delta(T_g \phi_u) + q^2 (T_g(u^2))(T_g \phi_u) = q\omega T_g(u^2).$$

But  $T_g(u^2) = (T_g u)^2$  and  $T_g u = u$  if  $u \in \hat{H}_\#^1(\mathbb{R}^3)$ , so that

$$-\Delta(T_g \phi_u) + q^2 u^2 T_g \phi_u = q\omega u^2.$$

By uniqueness (see Proposition 2.8), we have  $T_g \phi_u = \phi_u$ , so that  $\phi_u \in \mathcal{D}_\#^1(\mathbb{R}^3)$ .  $\square$

By the lemma above, we can define the map

$$u \in \hat{H}_\#^1(\mathbb{R}^3) \mapsto Z_\omega(u) = \phi_u \in \mathcal{D}_\#^1 \text{ solution of (2.46)}. \quad (2.48)$$

Since  $\phi_u$  solves (2.46), clearly we have

$$d_\phi J(u, Z_\omega(u), \mathbf{A}) = 0, \quad (2.49)$$

where  $J$  is defined in (2.22) and  $d_\phi J$  denotes the partial differential of  $J$  with respect to  $\phi$ .

Following the lines of the proof of [11, Proposition 2.1], using Lemma 2.9, we can easily prove the following result:

**Proposition 2.10.** *The map  $Z_\omega$  defined in (2.48) is of class  $C^1$  and*

$$(Z'_\omega[u])[v] = 2q (\Delta - q^2 u^2)^{-1} [(q\phi_u - \omega)uv] \quad \forall u, v \in \mathcal{D}_\#^1. \quad (2.50)$$

For  $u \in H^1(\mathbb{R}^3)$ , let  $\Phi = \Phi_u$  be the solution of (2.46) with  $\omega = 1$ ; then  $\Phi$  solves the equation

$$-\Delta\Phi_u + q^2 u^2 \Phi_u = qu^2, \quad (2.51)$$

and clearly

$$\phi_u = \omega \Phi_u. \quad (2.52)$$

Now let  $q > 0$ ; then, by maximum principle arguments, one can show that for any  $u \in H^1(\mathbb{R}^3)$  the solution  $\Phi_u$  of (2.51) satisfies the following estimate, first proved in [24],

$$0 \leq \Phi_u \leq \frac{1}{q}. \quad (2.53)$$

Now, if  $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^1)^3$ , we introduce the *reduced action functional*

$$\tilde{J}(u, \mathbf{A}) = J(u, Z_\omega(u), \mathbf{A}).$$

Recalling that  $J$  and the map  $u \rightarrow Z_\omega(u) = \phi_u$  are of class  $C^1$  by Lemma 2.5 and Proposition 2.10), respectively, also the functional  $\tilde{J}$  is of class  $C^1$ . Now, by using the chain rule and (2.49), it is standard to show that the following Lemma holds:

**Lemma 2.11.** *If  $(u, \mathbf{A})$  is a critical point of  $\tilde{J}$ , then  $(u, Z_\omega(u), \mathbf{A})$  is a critical point of  $J$  (and viceversa).*

From (2.51) we have

$$\int_{\mathbb{R}^3} q u^2 \Phi_u dx = \int_{\mathbb{R}^3} |\nabla \Phi_u|^2 dx + q^2 \int_{\mathbb{R}^3} u^2 \Phi_u^2 dx, \quad (2.54)$$

which is another way of writing (2.49).

Now, by (2.52) and (2.54), we have:

$$\begin{aligned} \tilde{J}(u, \mathbf{A}) &= J(u, Z_\omega(u), \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 - |\nabla \phi_u|^2 + |\nabla \times \mathbf{A}|^2] dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} [|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi_u)^2] u^2 dx + \int_{\mathbb{R}^3} W(u) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |l\nabla\theta - q\mathbf{A}|^2 u^2] dx + \int_{\mathbb{R}^3} W(u) dx \\ &\quad - \frac{\omega^2}{2} \int_{\mathbb{R}^3} (1 - q\Phi_u) u^2 dx. \end{aligned}$$

Then

$$\tilde{J}(u, \mathbf{A}) = I(u, \mathbf{A}) - \frac{\omega^2}{2} K_q(u), \quad (2.55)$$

where  $I : \hat{H}^1 \times (\mathcal{D}^1)^3 \rightarrow \mathbb{R}$  and  $K_q : \hat{H}^1 \rightarrow \mathbb{R}$  are defined as

$$I(u, \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |l\nabla\theta - q\mathbf{A}|^2 u^2) dx + \int_{\mathbb{R}^3} W(u) dx \quad (2.56)$$

and

$$K_q(u) = \int_{\mathbb{R}^3} (1 - q\Phi_u) u^2 dx. \quad (2.57)$$

Now, let us introduce the *reduced energy functional*, defined as

$$\hat{\mathcal{E}}(u, \mathbf{A}) = \mathcal{E}(u, Z_\omega(u), \mathbf{A}),$$

where, as in (2.15),

$$\begin{aligned} \mathcal{E}(u, \phi, \mathbf{A}) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 + (|l\nabla\theta - q\mathbf{A}|^2 + (\omega - q\phi)^2) u^2) dx \\ &\quad + \int_{\mathbb{R}^3} W(u) dx. \end{aligned} \quad (2.58)$$



By using (2.54) and (2.52), we easily find that

$$\hat{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u). \quad (2.59)$$

Recalling (2.16) and (2.17), we note that

$$Q = q\sigma = q\omega K_q(u)$$

represents the (electric) charge, so that, if  $u \neq 0$ , we can write

$$\hat{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)}.$$

Then for any  $\sigma \neq 0$ , the functional  $E_{\sigma,q} : (\hat{H}^1 \setminus \{0\}) \times (\mathcal{D}^1)^3 \rightarrow \mathbb{R}$ , defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)} \quad (2.60)$$

represents the energy on the configuration  $(u, \omega\Phi_u, \mathbf{A})$  having charge  $Q = q\sigma$  or, equivalently, frequency  $\omega = \frac{\sigma}{K_q(u)}$ .

The following lemma holds (see [5, Lemma 13]):

**Lemma 2.12.** *The functional*

$$\hat{H}^1 \ni u \mapsto K(u) = \int_{\mathbb{R}^3} (1 - q\Phi_u)u^2 dx$$

*is differentiable and for any  $u, v \in \hat{H}^1$  we have*

$$K'(u)[v] = 2 \int_{\mathbb{R}^3} (1 - q\Phi_u)^2 uv dx. \quad (2.61)$$

Introducing  $E_{\sigma,q}$  turns out to be a useful choice, as the following easy consequence shows (see [5, Proposition 14]):

**Proposition 2.13.** *Let  $\sigma \neq 0$  and let  $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^1)^3$ ,  $u \neq 0$  be a critical point of  $E_{\sigma,q}$ . Then, if we set  $\omega = \frac{\sigma}{K_q(u)}$ ,  $(u, Z_\omega(u), \mathbf{A})$  is a critical point of  $J$ .*

Therefore, by Proposition 2.13 and Theorem 2.6 we are reduced to study the critical points of  $E_{\sigma,q}$ , which is a functional bounded from below, since all its components are nonnegative.

However  $E_{\sigma,q}$  contains the term  $\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2$ , which is not a Sobolev norm in  $(\mathcal{D}^1)^3$ . In order to avoid consequent difficulties, we introduce a suitable manifold  $V \subset \hat{H}^1 \times (\mathcal{D}^1)^3$  in the following way: first, we set

$$\mathcal{A}_0 := \{\mathbf{X} \in C_C^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3) : \mathbf{X} = b(r, z)\nabla\theta; \ b \in C_C^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R})\},$$

and we denote by  $\mathcal{A}$  the closure of  $\mathcal{A}_0$  with respect to the norm of  $(\mathcal{D}^1)^3$ . We now consider the space

$$V := \hat{H}_\#^1 \times \mathcal{A},$$

and we set  $U = (u, \mathbf{A}) \in V$  with

$$\|U\|_V = \|(u, \mathbf{A})\|_V = \|u\|_{\hat{H}_\#^1} + \|\mathbf{A}\|_{(\mathcal{D}^1)^3}.$$

We need the following result, for whose proof see [5, Lemma 15]:

**Lemma 2.14.** *If  $\mathbf{A} \in \mathcal{A}$ , then*

$$\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx.$$

Working in  $V$  has two advantages: first, the components  $\mathbf{A}$  of the elements in  $V$  are divergence free, so that the term  $\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2$  can be replaced by  $\|\mathbf{A}\|_{(\mathcal{D}^1)^3}^2 = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2$ . Second, the critical points of  $J$  constrained on  $V$  satisfy system (1.5)–(1.6)–(1.7); namely  $V$  is a “natural constraint” for  $J$ .

### 3 Proof of Theorem 1.2

In this section we shall always assume that  $W$  satisfies  $W1)$ ,  $W2)$ ,  $W3)$  and we will show that  $E_{\sigma,q}$  constrained on  $V$  has a minimum which is a nontrivial solution of system (1.5)–(1.6)–(1.7).

We start with the following *a priori* estimate on minimizing sequences:

**Lemma 3.1.** *For any  $\sigma, q > 0$ , any minimizing sequence  $(u_n, \mathbf{A}_n) \subset V$  for  $E_{\sigma,q}|_V$  is bounded in  $\hat{H}^1 \times (\mathcal{D}^1)^3$ .*

*Proof.* It is similar to the proof of [5, Lemma 18], so we only sketch it.

Let  $(u_n, \mathbf{A}_n) \subset V$  be a minimizing sequence for  $E_{\sigma,q}|_V$ . Clearly, by definition of  $E_{\sigma,q}$ , see (2.60), we get that  $\|\mathbf{A}_n\|_{(\mathcal{D}^1)^3}$  is bounded.

Then, we show that  $\|u_n\|_{L^2}$  is bounded; indeed, since  $(u_n, \mathbf{A}_n)$  is a minimizing sequence for  $E_{\sigma,q}|_V$  we get that  $\int_{\mathbb{R}^3} W(u_n) dx$  and  $\int_{\mathbb{R}^3} |\nabla u_n|^2 dx$  are bounded. Then, thanks to the Sobolev Embedding Theorem, we also have that

$$\int_{\mathbb{R}^3} u_n^6 dx \quad \text{is bounded.} \tag{3.62}$$

Let  $\epsilon > 0$  and set

$$\Omega_n = \{x \in \mathbb{R}^3 : |u_n(x)| > \epsilon\} \quad \text{and} \quad \Omega_n^c = \mathbb{R}^3 \setminus \Omega_n.$$

Since  $W \geq 0$ , by the very definition of  $E_{\sigma,q}$ , we have that also  $\int_{\Omega_n^c} W(u_n) dx$  is bounded. By  $W2)$  we can write

$$W(s) = \frac{m^2}{2} s^2 + o(s^2).$$

Then, if  $\epsilon$  is small enough, there is a constant  $c > 0$  such that

$$\int_{\Omega_n^c} W(u_n) dx \geq c \int_{\Omega_n^c} u_n^2 dx,$$

and so

$$\int_{\Omega_n^c} u_n^2 dx \text{ is bounded.}$$

In conclusion, we have that

$$\epsilon^6 \text{meas}(\Omega_n) \leq \int_{\Omega_n} u_n^6 dx + \int_{\Omega_n^c} u_n^6 dx = \int_{\mathbb{R}^3} u_n^6 dx \leq c \Rightarrow \text{meas}(\Omega_n) \text{ is bounded.} \quad (3.63)$$

On the other hand, thanks to Hölder's inequality, we have

$$\int_{\Omega_n} u_n^2 dx \leq \left( \int_{\Omega_n} u_n^6 dx \right)^{\frac{1}{3}} \text{meas}(\Omega_n)^{\frac{2}{3}}. \quad (3.64)$$

By (3.62), (3.64) and (3.63), we finally get that

$$\int_{\Omega_n} u_n^2 dx \text{ is bounded,}$$

So that  $(u_n)_n$  is bounded in  $L^2(\mathbb{R}^3)$ .

The fact that  $(u_n)_n$  is bounded in  $\hat{H}_\#^1(\mathbb{R}^3)$  follows as in [5].  $\square$

**Proposition 3.2.** *For any  $\sigma, q > 0$  there exists a minimizing sequence  $U_n = (u_n, \mathbf{A}_n)$  of  $E_{\sigma,q}|_V$ , with  $u_n \geq 0$  and which is also a Palais–Smale sequence for  $E_{\sigma,q}$ , i.e.*

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow 0.$$

*Proof.* Let  $(u_n, \mathbf{A}_n) \subset V$  be a minimizing sequence for  $E_{\sigma,q}|_V$ . It is not restrictive to assume that  $u_n \geq 0$ . Otherwise, we can replace  $u_n$  with  $|u_n|$  and we still have a minimizing sequence (see (2.58)). By Ekeland's Variational Principle (see [14]) we can also assume that  $(u_n, \mathbf{A}_n)$  is a Palais–Smale sequence for  $E_{\sigma,q}|_V$ , namely we can assume that

$$E'_{\sigma,q}|_V(u_n, \mathbf{A}_n) \rightarrow 0.$$

By using the same technique used to prove Theorem 16 in [6], it follows that  $(u_n, \mathbf{A}_n)$  is a Palais–Smale sequence also for  $E_{\sigma,q}$ , that is

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow 0.$$

$\square$

A fundamental tool in proving the existence result, is given by the following

**Lemma 3.3.** *For any  $\sigma, q > 0$  and for any minimizing sequence  $(u_n, \mathbf{A}_n) \subset V$  for  $E_{\sigma,q}|_V$ , there exist positive numbers  $a_1 < a_2$  such that*

$$a_1 \leq \int_{\mathbb{R}^3} (1 - q\Phi_{u_n}) u_n^2 dx \leq a_2 \text{ for every } n \in \mathbb{N}$$

and

$$a_1 \leq \int_{\mathbb{R}^3} u_n^2 dx \leq a_2 \text{ for every } n \in \mathbb{N}.$$

*Proof.* The upper bounds are an obvious consequence of Lemma 3.1 and of (2.53), so that we only prove the lower bounds.

Since  $E_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow \inf_V E_{\sigma,q}$ , from (2.60) we immediately get that there exists  $a_1 > 0$  such that

$$\frac{1}{\int_{\mathbb{R}^3} (1 - q\Phi_{u_n}) u_n^2 dx} \leq \frac{1}{a_1} \text{ for every } n \in \mathbb{N},$$

and thus all the claims follow.  $\square$

As a corollary of the previous Lemma, whose proof is now very easy, but whose consequences are crucial:

**Lemma 3.4.** *For any  $\sigma, q > 0$*

$$\inf_V E_{\sigma,q} > 0.$$

*Proof.* Assume by contradiction that  $\inf_V E_{\sigma,q} = 0$ . Hence, there exists a sequence  $(u_n, \mathbf{A}_n)_n \subset V$  such that  $E_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since both  $I$  and  $K_q$  are nonnegative, from (2.60) we get

$$I(u_n, \mathbf{A}_n) \rightarrow 0 \text{ and } \frac{1}{K_q(u_n)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular,

$$\int_{\mathbb{R}^3} (1 - q\Phi_{u_n}) u_n^2 dx \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and thus, by (2.53),

$$\int_{\mathbb{R}^3} u_n^2 dx \rightarrow \infty \text{ as } n \rightarrow \infty,$$

a contradiction with Lemma 3.3.  $\square$

The following result, which turns out to be a crucial one, is the only point where assumption W4) is used.

**Lemma 3.5.** *For all  $\sigma, q > 0$  there exists  $u_0 \in \hat{H}^1$  such that*

$$E_{\sigma,q}(u_0, 0) < m\sigma.$$

*Proof.* Let us define

$$v(x) := \begin{cases} 1 - \sqrt{(r-2)^2 + x_3^2}, & (r-2)^2 + x_3^2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We define the set  $A_\lambda := \{(r, x_3) \in \mathbb{R}^3 \text{ s.t. } (r-2\lambda)^2 + x_3^2 \leq \lambda^2\}$  and we compute

$$|A_\lambda| = \int_{A_\lambda} dx_1 dx_2 dx_3 = 4\pi^2 \lambda^3 = \lambda^3 |A_1|. \quad (3.65)$$

Of course,  $v \in \hat{H}_r^1$  and, for a future need, we also compute

$$\begin{aligned} \int_{\mathbb{R}^3} v^2 dx &= \int_{A_1} \left(1 - \sqrt{(r-2)^2 + x_3^2}\right)^2 dx_1 dx_2 dx_3 = \frac{2}{3} \pi^2, \\ \int_{\mathbb{R}^3} v dx &= \int_{A_1} \left(1 - \sqrt{(r-2)^2 + x_3^2}\right) dx_1 dx_2 dx_3 = \frac{4}{3} \pi^2, \\ \int_{\mathbb{R}^3} |\nabla v|^2 dx &= \int_{A_1} dx_1 dx_2 dx_3 = 4\pi^2. \end{aligned} \quad (3.66)$$

Moreover, for  $\varepsilon \in (0, \varepsilon_0)$  and  $\lambda \geq 1$  we define

$$u_{\varepsilon, \lambda}(x) = \varepsilon^2 \lambda v\left(\frac{x}{\lambda}\right).$$

We also choose  $\varepsilon$  and  $\lambda$  such that

$$\varepsilon \lambda \leq 1, \quad (3.67)$$

so that  $0 \leq u_{\varepsilon, \lambda} \leq \varepsilon < \varepsilon_0$  in  $\mathbb{R}^3$ .

Then we have

$$\begin{aligned} E_{\sigma_\lambda, q}(u_{\varepsilon, \lambda}, 0) &= \int_{\mathbb{R}^3} \left[ \frac{1}{2} |\nabla u_{\varepsilon, \lambda}|^2 + \frac{l^2}{r^2} \frac{u_{\varepsilon, \lambda}^2}{2} + W(u_{\varepsilon, \lambda}) \right] dx + \frac{\sigma^2}{2K_q(u_{\varepsilon, \lambda})} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_{\varepsilon, \lambda}|^2 + \frac{l^2}{2} \int_{\mathbb{R}^3} \frac{u_{\varepsilon, \lambda}^2}{r^2} + \frac{m^2}{2} \int_{\mathbb{R}^3} u_{\varepsilon, \lambda}^2 + \int_{\mathbb{R}^3} N(u_{\varepsilon, \lambda}) dx + \frac{\sigma^2}{2K_q(u_{\varepsilon, \lambda})}. \end{aligned} \quad (3.68)$$

Now, observe that in  $A_\lambda$  we have

$$r \geq 2\lambda - \sqrt{\lambda^2 - x_3^2} \geq \lambda,$$

so that, thanks to (3.65), we can estimate

$$\begin{aligned}
 \int_{\mathbb{R}^3} \frac{u_{\varepsilon,\lambda}^2}{r^2} dx_1 dx_2 dx_3 &= \varepsilon^4 \int_{A_\lambda} \frac{\left( \lambda - \lambda \sqrt{\left( \frac{r}{\lambda} - 2 \right)^2 + \frac{x_3^2}{\lambda^2}} \right)^2}{r^2} dr dx_3 \\
 &\leq \varepsilon^4 \int_{A_\lambda} \frac{\left( \lambda - \sqrt{(r-2\lambda)^2 + x_3^2} \right)^2}{\lambda^2} dr dx_3 \\
 &\leq \varepsilon^4 \int_{A_\lambda} \left( \frac{\lambda - \sqrt{(r-2\lambda)^2 + x_3^2}}{\lambda} \right)^2 dr dx_3 \leq \varepsilon^4 |A_\lambda| = 4\pi^2 \varepsilon^4 \lambda^3.
 \end{aligned} \tag{3.69}$$

By the change of variable  $y = x/\lambda$  we immediately get

$$\begin{aligned}
 \int_{A_\lambda} |\nabla u_{\varepsilon,\lambda}|^2 dx &= \varepsilon^4 \lambda^3 \int_{A_1} |\nabla v|^2 dx, \\
 \int_{A_\lambda} (u_{\varepsilon,\lambda})^\vartheta dx &= \varepsilon^{2\theta} \lambda^{\vartheta+3} \int_{A_1} v^\vartheta dx \quad \forall \theta > 0.
 \end{aligned}$$

Therefore, (3.66), (3.68), (3.69) and W4) imply

$$\begin{aligned}
 E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) &\leq 2\pi^2 \varepsilon^4 \lambda^3 + \frac{m^2 \pi^2}{3} \varepsilon^4 \lambda^5 + 2\pi^2 l^2 \varepsilon^4 \lambda^3 \\
 &\quad - D \varepsilon^{2\tau} \lambda^{\tau+3} \int_{A_1} v^\tau dx + \frac{\sigma^2}{2K_q(u_{\varepsilon,\lambda})}.
 \end{aligned} \tag{3.70}$$

Now, let us note that

$$-\Delta \Phi_{u_{\varepsilon,\lambda}} = q u_{\varepsilon,\lambda}^2 (1 - q \Phi_{u_{\varepsilon,\lambda}}) \leq q u_{\varepsilon,\lambda}^2,$$

so that, by the Comparison Principle, for every  $x \in \mathbb{R}^3$  we have

$$\Phi_{u_{\varepsilon,\lambda}}(x) \leq \frac{q}{4\pi} \int_{\mathbb{R}^3} \frac{u_{\varepsilon,\lambda}^2(x-y)}{|y|} dy = \frac{q \varepsilon^4 \lambda^5}{4\pi} \int_{\mathbb{R}^3} \frac{v^2(y)}{|x-\lambda y|} dy \leq \frac{q}{2} \varepsilon^4 \lambda^4. \tag{3.71}$$

Indeed:

$$\begin{aligned}
 \int_{\mathbb{R}^3} \frac{v^2(y)}{|x-\lambda y|} dy &\leq \int_{A_1} \frac{1}{|x-\lambda y|} dy = \frac{1}{\lambda^3} \int_{A_{1/\lambda}} \frac{1}{|x-z|} dz \\
 &= \frac{1}{\lambda^3} \int_{A_{1/\lambda}-x} \frac{1}{|z|} dz \leq \frac{1}{\lambda^3} \int_{B(0,1/\lambda)} \frac{1}{|z|} dz = \frac{2\pi}{\lambda},
 \end{aligned}$$

and (3.71) follows.

As a consequence,

$$\begin{aligned}
 K_q(u_{\varepsilon,\lambda}) &= \int_{\mathbb{R}^3} u_{\varepsilon,\lambda}^2 (1 - q \Phi_{u_{\varepsilon,\lambda}}) dx \geq \int_{\mathbb{R}^3} u_{\varepsilon,\lambda}^2 (1 - \frac{q^2}{2} \varepsilon^4 \lambda^4) dx \\
 &= \frac{2}{3} \pi^2 (1 - \frac{q^2}{2} \varepsilon^4 \lambda^4) \varepsilon^4 \lambda^5.
 \end{aligned}$$

Hence, choosing

$$\varepsilon^4 \lambda^4 < 2/q^2, \quad (3.72)$$

(3.70) becomes

$$\begin{aligned} E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) &\leq 2\pi^2 \varepsilon^4 \lambda^3 + \frac{m^2 \pi^2}{3} \varepsilon^4 \lambda^5 + 2\pi^2 l^2 \varepsilon^4 \lambda^3 \\ &\quad - D \varepsilon^{2\tau} \lambda^{\tau+3} \int_{A_1} v^\tau dx + \frac{3\sigma^2}{2\pi^2(1 - \frac{q^2}{2} \varepsilon^4 \lambda^4) \varepsilon^4 \lambda^5}. \end{aligned}$$

Now, take

$$\varepsilon^4 \lambda^5 = \frac{6\sigma}{m\pi^2},$$

so that

$$E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) \leq 12 \frac{\sigma}{m} (1 + l^2) \lambda^{-2} + 2m\sigma - E \lambda^{3-3\tau/2} + \frac{m\sigma}{4(1 - \frac{3\sigma q^2}{\lambda m \pi^2})},$$

where we have set  $E = D(6\sigma/m\pi^2)^{\tau/2} \int v^\tau$ .

Since  $\lambda \geq 1$  and  $\varepsilon < 1$  we can also assume that  $\lambda \geq 6q^2\sigma/m\pi^2$ , so that

$$E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) \leq 12 \frac{\sigma}{m} (1 + l^2) \lambda^{-2} + 2m\sigma - E \lambda^{3-3\tau/2} + \frac{m\sigma}{2}.$$

Now, if  $D$  is sufficiently large, we can find  $\lambda \geq \max\{1, 6q^2\sigma/m\pi^2\}$  and satisfying (3.67) and (3.72) such that

$$12 \frac{\sigma}{m} (1 + l^2) \lambda^{-2} + \frac{5}{2} m\sigma - E \lambda^{3-3\tau/2} \leq m\sigma,$$

that is

$$12 \frac{\sigma}{m} (1 + l^2) + \frac{3}{2} m\sigma \lambda^2 - E \lambda^{5-3\tau/2} \leq 0.$$

For example, if  $\tau > 10/3$ , it will be enough to choose  $\lambda$  such that

$$\lambda \leq \left( \frac{E}{12 \frac{\sigma}{m} (1 + l^2) + \frac{3}{2} m\sigma} \right)^{2/(3\tau-10)},$$

provided that the right hand side is so large that (3.67) and (3.72) hold true as well. If, on the contrary,  $\tau \in (2, 10/3)$ , one can take

$$\lambda \leq \sqrt{\frac{E - 12 \frac{\sigma}{m} (1 + l^2)}{\frac{3}{2} m\sigma}}.$$

□

Then we can prove the following

**Lemma 3.6.** *For all  $\sigma, q > 0$  there exists  $c > 0$  and a minimizing sequence  $U_n = (u_n, \mathbf{A}_n) \subset V$  of  $E_{\sigma,q}|_V$  such that*

$$\int_{\mathbb{R}^3} (|u_n|^\ell + |u_n|^p) dx \geq c > 0 \text{ for every } n \in \mathbb{N}.$$

*Proof.* By Lemma 3.5 we know that there exists  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that

$$E_{\sigma,q}(u_n, \mathbf{A}_n) \leq m\sigma - \delta,$$

which implies in particular that

$$\frac{m^2}{2} \int_{\mathbb{R}^3} u_n^2 dx + \int_{\mathbb{R}^3} N(u_n) dx + \frac{\sigma^2}{2 \int_{\mathbb{R}^3} u_n^2 dx} \leq m\sigma - \delta.$$

Thus

$$\int_{\mathbb{R}^3} N(u_n) dx \leq m\sigma - \delta - \left( \frac{m^2}{2} \int_{\mathbb{R}^3} u_n^2 dx + \frac{\sigma^2}{2 \int_{\mathbb{R}^3} u_n^2 dx} \right) \leq -\delta,$$

since  $a/(2b) + b/(2a) \geq 1$  for any  $a, b > 0$ . Then

$$\left| \int_{\mathbb{R}^3} N(u_n) dx \right| \geq \delta \quad \text{for all } n \geq n_0,$$

and  $W2)$  imply the claim, up to a relabelling of the sequence.  $\square$

By Lemma 3.1 we know that any minimizing sequence  $U_n := (u_n, \mathbf{A}_n) \subset V$  of  $E_{\sigma,q}|_V$  weakly converges (up to a subsequence). Observe that  $E_{\sigma,q}$  is invariant for translations along the  $z$ -axis, namely for every  $U \in V$  and  $L \in \mathbb{R}$  we have

$$E_{\sigma,q}(T_L U) = E_{\sigma,q}(U),$$

where

$$T_L(U)(x, y, z) = U(x, y, z + L). \quad (3.73)$$

As a consequence of this invariance, we have that  $(u_n, \mathbf{A}_n)$  does not contain in general a strongly convergent subsequence. To overcome this difficulty, we will show that there exists a minimizing sequence  $(u_n, \mathbf{A}_n)$  of  $E_{\sigma,q}|_V$  which, up to translations along the  $z$ -direction, weakly converges to a non-trivial limit  $(u_0, \mathbf{A}_0)$ . Eventually, we will show that  $(u_0, \mathbf{A}_0)$  is a critical point of  $E_{\sigma_0,q}$  for some charge  $\sigma_0$ .

In order to proceed with this project, we start proving the following weak compactness result, whose proof is an adaptation of [5, Proposition 22], but whose statement is much more general:

**Proposition 3.7.** *For any  $\sigma, q > 0$  there exists a Palais-Smale sequence  $U_n = (u_n, \mathbf{A}_n)$  of  $E_{\sigma,q}$  which weakly converges to  $(u_0, \mathbf{A}_0)$ ,  $u_0 \geq 0$  and  $u_0 \not\equiv 0$ .*



*Proof.* By Proposition 3.2, we know that there exists a minimizing sequence  $U_n = (u_n, \mathbf{A}_n)$  of  $E_{\sigma,q}|_V$ , with  $u_n \geq 0$  and which is also a Palais–Smale sequence for  $E_{\sigma,q}$ . Moreover, by Lemma 3.6, we know that there exists  $c > 0$  such that

$$\|u_n\|_{L^\ell}^\ell + \|u_n\|_{L^p}^p \geq c > 0 \text{ for } n \text{ large.} \quad (3.74)$$

By Lemma 3.1 the sequence  $\{U_n\}$  is bounded in  $\hat{H}^1 \times (\mathcal{D}^1)^3$ , so we can assume that it weakly converges. However the weak limit could be trivial. We will show that there is a sequence of integers  $j_n$  such that  $V_n := T_{j_n} U_n \rightharpoonup U_0 = (u_0, \mathbf{A}_0)$  in  $H^1 \times (\mathcal{D}^1)^3$ , with  $u_0 \neq 0$ , see (3.73).

For any integer  $j$  we set

$$\Omega_j = \{(x_1, x_2, x_3) : j \leq x_3 < j+1\}.$$

In the following we denote by  $c$  various positive absolute constants which may vary also from line to line. We have for all  $n$ ,

$$\begin{aligned} \|u_n\|_{L^\ell}^\ell &= \sum_j \left( \int_{\Omega_j} |u_n|^\ell dx \right)^{1/\ell} \left( \int_{\Omega_j} |u_n|^\ell dx \right)^{\frac{\ell-1}{\ell}} \\ &\leq \sup_j \|u_n\|_{L^\ell(\Omega_j)} \sum_j \left( \int_{\Omega_j} |u_n|^\ell dx \right)^{\frac{\ell-1}{\ell}} \\ &\leq c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \sum_j \|u_n\|_{H^1(\Omega_j)}^{\ell-1} \\ &= c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \|u_n\|_{H^1(\mathbb{R}^3)}^{\ell-1} \leq (\text{since } \|u_n\|_{H^1(\mathbb{R}^3)} \text{ is bounded}) \\ &\leq c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \quad \text{for all } n \geq 1. \end{aligned} \quad (3.75)$$

In the same way we get

$$\|u_n\|_{L^p}^p \leq c \sup_j \|u_n\|_{L^p(\Omega_j)} \quad \text{for all } n \geq 1. \quad (3.76)$$

Then, by (3.74), (3.75) and (3.76) it immediately follows that, for  $n$  large, we can choose an integer  $j_n$  such that

$$\|u_n\|_{L^\ell(\Omega_{j_n})} + \|u_n\|_{L^p(\Omega_{j_n})} \geq c > 0. \quad (3.77)$$

Now set

$$(u'_n, \mathbf{A}'_n) = U'_n(x_1, x_2, x_3) = T_{j_n}(U_n) = U_n(x_1, x_2, x_3 + j_n).$$

Since  $(U'_n)_n$  is again a minimizing sequence for  $E_{\sigma,q}|_V$ , by Lemma 3.1 the sequence  $\{u'_n\}$  is bounded in  $\hat{H}^1(\mathbb{R}^3)$ ; then (up to a subsequence) it weakly converges to  $u_0 \in \hat{H}^1(\mathbb{R}^3)$ . Clearly  $u_0 \geq 0$ , since  $u'_n \geq 0$ . We want to show that  $u_0 \neq 0$ . Now, let  $\varphi = \varphi(x_3)$  be a nonnegative,  $C^\infty$ -function whose value is

1 for  $0 < x_3 < 1$  and 0 for  $|x_3| > 2$ . Then, the sequence  $\varphi u'_n$  is bounded in  $H_0^1(\mathbb{R}^2 \times (-2, 2))$ , and moreover  $\varphi u'_n$  has cylindrical symmetry. Then, using the compactness result of Esteban–Lions [16], we have that, up to a subsequence,

$$\varphi u'_n \rightarrow \varphi u_0 \quad \text{in } L^\ell(\mathbb{R}^2 \times (-2, 2)), \quad \text{in } L^p(\mathbb{R}^2 \times (-2, 2)) \text{ and a.e. in } \mathbb{R}^2 \times (-2, 2). \quad (3.78)$$

Moreover for  $r = p, \ell$  we clearly have

$$\|\varphi u'_n\|_{L^r(\mathbb{R}^2 \times (-2, 2))} \geq \|u'_n\|_{L^r(\Omega_0)} = \|u_n\|_{L^r(\Omega_{j_n})}. \quad (3.79)$$

Then by (3.78), (3.79) and (3.77) we have

$$\|\varphi u_0\|_{L^\ell(\mathbb{R}^2 \times (-2, 2))} + \|\varphi u_0\|_{L^p(\mathbb{R}^2 \times (-2, 2))} \geq c > 0.$$

Thus we have that  $u_0 \neq 0$ , as claimed.  $\square$

Now we need the following Proposition which reminds [5, Proposition 23]. However, the statement we can prove is more general, and also the last part of the proof is different from the corresponding one.

**Proposition 3.8.** *For every  $q > 0$  there exists  $\sigma_0 > 0$  such that  $E_{\sigma_0, q}$  has a critical point  $(u_0, \mathbf{A}_0)$ ,  $u_0 \neq 0$ ,  $u_0 \geq 0$ .*

*Proof.* By Proposition 3.7, there exists a sequence  $U_n = (u_n, \mathbf{A}_n)$  in  $V$ , with  $u_n \geq 0$  and such that

$$E'_{\sigma, q}(u_n, \mathbf{A}_n) \rightarrow 0 \quad (3.80)$$

and

$$(u_n, \mathbf{A}_n) \rightharpoonup (u_0, \mathbf{A}_0), \quad u_0 \geq 0, \quad u_0 \neq 0.$$

We now show that there exists a charge  $\sigma_0 > 0$  such that  $U_0 = (u_0, \mathbf{A}_0)$  is a critical point of  $E_{\sigma_0, q}$ .

By (3.80), in particular we get that

$$dE_{\sigma, q}(U_n)[w, 0] \rightarrow 0 \quad \text{and} \quad dE_{\sigma, q}(U_n)[0, \mathbf{w}] \rightarrow 0, \quad \text{for any } (w, \mathbf{w}) \in \hat{H}^1 \times (C_C^\infty)^3.$$

Then for any  $w \in \hat{H}^1$  and  $\mathbf{w} \in (C_C^\infty)^3$  we have

$$\partial_u I(U_n)[w] + \partial_u \left( \frac{\sigma^2}{2K_q(u_n)} \right) [w] \rightarrow 0 \quad (3.81)$$

and

$$\partial_{\mathbf{A}} I(U_n)[\mathbf{w}] \rightarrow 0, \quad (3.82)$$

where  $\partial_u$  and  $\partial_{\mathbf{A}}$  denote the partial derivatives of  $I$  with respect to  $u$  and  $\mathbf{A}$ , respectively. So from (3.81) we get for any  $w \in \hat{H}^1$ ,

$$\partial_u I(U_n)[w] - \frac{\sigma^2 K'_q(u_n)}{2(K_q(u_n))^2} [w] \rightarrow 0,$$

which can be written as follows:

$$\partial_u I(U_n)[w] - \frac{\omega_n^2 K'_q(u_n)}{2}[w] \rightarrow 0, \quad (3.83)$$

where we have set

$$\omega_n = \frac{\sigma}{K_q(u_n)}.$$

By Lemma 3.3 we have that (up to a subsequence)

$$\omega_n \rightarrow \omega_0 > 0.$$

Then by (3.83) we get for any  $w \in \hat{H}^1$

$$\partial_u I(U_n)[w] - \frac{\omega_0^2 K'_q(u_n)}{2}[w] \rightarrow 0. \quad (3.84)$$

Now, let  $\Phi_n$  be the solution in  $\mathcal{D}^1$  of the equation

$$-\Delta \Phi_n + q^2 u_n^2 \Phi_n = q u_n^2. \quad (3.85)$$

Since  $\{u_n\}$  is bounded in  $H^1$  and since  $\Phi_n$  solves (3.85), by (2.47) we have that  $\{\Phi_n\}$  is bounded in  $\mathcal{D}^1$  and, checking with test functions in  $C_c^\infty(\mathbb{R}^3)$ , it is easy to see that (up to a subsequence) its weak limit  $\Phi_0$  is a weak solution of

$$-\Delta \Phi_0 + q^2 u_0^2 \Phi_0 = q u_0^2. \quad (3.86)$$

Moreover, by Lemma 2.12, we have

$$K'_q(u_n)[w] = 2 \int_{\mathbb{R}^3} u_n w (1 - q \Phi_n)^2 dx \quad \text{and} \quad K'_q(u_0)[w] = 2 \int_{\mathbb{R}^3} u_0 w (1 - q \Phi_0)^2 dx \quad (3.87)$$

for every  $w \in \hat{H}^1$ .

We claim that

$$K'_q(u_n)[w] \rightarrow K'_q(u_0)[w] \quad \text{for any } w \in \hat{H}^1. \quad (3.88)$$

Indeed, by (2.21), for any  $w \in \hat{H}^1$  and every  $\varepsilon > 0$ , there exists  $w_\varepsilon \in C_c^\infty \cap \hat{H}^1$  such that  $\|w - w_\varepsilon\|_{\hat{H}^1} < \varepsilon$ . Then,

$$\begin{aligned} K'_q(u_n)[w] - K'_q(u_0)[w] &= K'_q(u_n)[w - w_\varepsilon] \\ &\quad + [K'_q(u_n) - K'_q(u_0)][w_\varepsilon] - K'_q(u_0)[w_\varepsilon - w]. \end{aligned}$$

But the sequence of operators  $(K'_q(u_n))_n$  is bounded in  $(\hat{H}^1)'$ , while  $[K'_q(u_n) - K'_q(u_0)][w_\varepsilon] \rightarrow 0$  by the Rellich Theorem. The claim follows.

Similar estimates show that for any  $w \in \hat{H}^1$

$$\partial_u I(U_n)[w] \rightarrow \partial_u I(U_0)[w]. \quad (3.89)$$

Then, passing to the limit in (3.84), by (3.88) and (3.89), we get

$$\partial_u I(U_0)[w] - \frac{\omega_0^2 K'_q(u_0)}{2}[w] = 0 \text{ for any } w \in \hat{H}^1. \quad (3.90)$$

On the other hand, similar arguments show that we can pass to the limit also in  $\partial_{\mathbf{A}} I(U_n)[\mathbf{w}]$  and have

$$\partial_{\mathbf{A}} I(U_n)[\mathbf{w}] \rightarrow \partial_{\mathbf{A}} I(U_0)[\mathbf{w}] \text{ for all } \mathbf{w} \in (C_C^\infty)^3. \quad (3.91)$$

From (3.82) and (3.91) we get

$$\partial_{\mathbf{A}} I(U_0)[\mathbf{w}] = 0 \text{ for all } \mathbf{w} \in (C_C^\infty)^3,$$

and, by density, for any  $\mathbf{w} \in (\mathcal{D}^1)^3$ . From (3.90) we thus deduce that  $U_0 = (u_0, \mathbf{A}_0)$  is a critical point of  $E_{\sigma_0, q}$  with  $\sigma_0 = \omega_0 K_q(u_0) > 0$ .  $\square$

Now we are ready to prove the main existence Theorem 1.2.

*Proof of Theorem 1.2.* The first part of Theorem 1.2 immediately follows from Propositions 2.13, 3.8 and Theorem 2.6. In fact, if the couple  $(u_0, \mathbf{A}_0)$  is like in Proposition 3.8, by Proposition 2.13 and Theorem 2.6 we deduce that  $(u_0, \omega_0, \phi_0, \mathbf{A}_0)$  with  $\omega_0 = \frac{\sigma_0}{K_q(u_0)}$ ,  $\phi_0 = Z_{\omega_0}(u_0)$ , solves (1.5)–(1.6)–(1.7).

Now assume  $q = 0$ , then, by (1.6) and (1.7), we easily deduce that  $\phi_0 = 0$  and  $\mathbf{A}_0 = 0$ . Finally assume that  $q > 0$ . Then, since  $\omega_0 > 0$ , by (1.6) we deduce that  $\phi_0 \neq 0$ . Moreover by (1.7) we deduce that  $\mathbf{A}_0 \neq 0$  if and only if  $l \neq 0$ .  $\square$

## 4 Solutions with full probability

Throughout this section we are concerned with a different approach to system (1.5)–(1.6)–(1.7): namely, we look for solutions having full probability and we prove Proposition 1.10. From a physical point of view such solutions are the most relevant ones, and in general they cannot be obtained from the solutions found in Theorem 1.2 by a rescaling argument, unless some homogeneity in the potential is given. But this is not the case if  $N \neq 0$ .

Therefore, we will work in the new manifold  $\tilde{V} := V \cap \mathcal{S}$ , where

$$\mathcal{S} = \left\{ (u, \mathbf{A}) \in V : \int_{\mathbb{R}^3} u^2 dx = 1 \right\}.$$

We follow the lines of the previous part of the paper, and for this reason we will be sketchy, though some differences will appear. For example, we begin with the following

**Proposition 4.1.** *For any  $\sigma, q \geq 0$  there exists a minimizing sequence  $U_n = (u_n, \mathbf{A}_n)$  of  $E_{\sigma, q}|_{\tilde{V}}$ , with  $u_n \geq 0$ , and a sequence  $(\mu_n)_n \in \mathbb{R}$ , such that*

$$E'_{\sigma, q}(u_n, \mathbf{A}_n)(v, \mathbf{B}) - \mu_n \int_{\mathbb{R}^3} u_n v dx \rightarrow 0 \quad \forall (v, \mathbf{B}) \in \tilde{V}.$$

Moreover,  $(\mu_n)_n$  converges to some  $\mu \in \mathbb{R}$  as  $n \rightarrow \infty$ .

*Proof.* Let  $(u_n, \mathbf{A}_n) \subset V$  be a minimizing sequence for  $E_{\sigma,q}|_{\hat{V}}$ . Working with  $u_n \geq 0$ , or replacing  $u_n$  with  $|u_n|$  if necessary, we still have a minimizing sequence (see (2.58)). By Ekeland's Variational Principle we can also assume that  $(u_n, \mathbf{A}_n)$  is a Palais-Smale sequence for  $E_{\sigma,q}|_{\hat{V}}$ , namely we can assume that

$$E'_{\sigma,q}|_{\hat{V}}(u_n, \mathbf{A}_n) \rightarrow 0,$$

i.e. there exists a sequence  $(\mu_n)_n \in \mathbb{R}$  with

$$E'_{\sigma,q}(u_n, \mathbf{A}_n)(v, \mathbf{B}) - \mu_n \int_{\mathbb{R}^3} u_n v dx \rightarrow 0, \quad \forall v \in \hat{H}^1_{\#}, \quad \forall \mathbf{B} \in \mathcal{A}. \quad (4.92)$$

Taking  $(u_n, \mathbf{A}_n)$  as a test function and using  $\int_{\mathbb{R}^3} u_n^2 dx = 1$  for all  $n \in \mathbb{N}$ , we get

$$E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) - \mu_n \int_{\mathbb{R}^3} u_n^2 dx = E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) - \mu_n \rightarrow 0. \quad (4.93)$$

From (4.93) we get

$$\begin{aligned} \mu_n &= E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) + o(1) \\ &= \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}_n|^2 u_n^2 dx + \int_{\mathbb{R}^3} W'(u_n) u_n dx \\ &\quad + \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}_n|^2 dx + q \int_{\mathbb{R}^3} u_n^2 |\mathbf{A}_n|^2 dx + \sigma K'_q(u_n) u_n dx + o(1), \end{aligned} \quad (4.94)$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, since all the terms in the right-hand-side of (4.94) are bounded, as already shown for Lemma 3.1, we get that also  $(\mu_n)_n$  is bounded; hence, there exists  $\mu \in \mathbb{R}$  such that, up to a subsequence,  $\mu_n \rightarrow \mu$  as  $n \rightarrow \infty$ .  $\square$

Now we restate Proposition 3.7 which still holds in this case thanks to Proposition 4.1, hence we get

**Proposition 4.2.** *For all  $\sigma, q > 0$  there exists a Palais-Smale sequence  $U_n = (u_n, \mathbf{A}_n)$  of  $E_{\sigma,q}$  which weakly converges to  $(u_0, \mathbf{A}_0)$ ,  $u_0 \geq 0$  and  $u_0 \neq 0$ .*

In order to prove Proposition 1.10 we should just notice that the analogous of Proposition 3.8 still holds using Proposition 4.1 and Proposition 4.2. Hence, we just restate the result of Proposition 3.8 as follows:

**Proposition 4.3.** *For every  $q > 0$  there exists  $\sigma_0 > 0$  such that  $E_{\sigma_0,q}$  has a critical point  $(u_0, \mathbf{A}_0)$ ,  $u_0 \neq 0$ ,  $u_0 \geq 0$ .*

Finally, we conclude with the

*Proof of Proposition 1.10.* It is a natural consequence of what already proved, exactly as done for the proof of Theorem 1.2 in the previous section. Namely, since Proposition 4.3 holds by Proposition 4.1 and Proposition 4.2, we can conclude that our claim is true thanks to Propositions 4.3, 2.13 and Theorem 2.6.

Now, suppose that  $\omega^2 \leq m^2$  and  $N'(s)s \geq 0$ . Passing to the limit as  $n \rightarrow \infty$  in (4.92) with  $v = u_0$  and  $\mathbf{B} = \mathbf{0}$ , as in the proof of Proposition 3.8, we get

$$\int_{\mathbb{R}^3} [|\nabla u_0|^2 + |l\nabla\theta - q\mathbf{A}_0|^2 u_0^2 - (\omega - q\phi_{u_0})^2 u_0^2 + W'(u_0)u_0] dx = \mu \int_{\mathbb{R}^3} u_0^2 dx,$$

which can be written as

$$\begin{aligned} & \int_{\mathbb{R}^3} [|\nabla u_0|^2 + |l\nabla\theta - q\mathbf{A}|^2 u_0^2 + (m^2 - \omega^2) u_0^2 - (q\phi - 2\omega) u_0^2 q\phi_{u_0} \\ & + N'(u_0)u_0] dx = \mu \int_{\mathbb{R}^3} u_0^2 dx. \end{aligned}$$

Thanks to (2.52), (2.53) and to the hypotheses under consideration, we get  $\mu > 0$ , so that the effective mass (see Definition 1.11) is strictly less than the original mass.  $\square$

## 5 Non-existence of standing solutions

In this section we shall prove Theorem 1.7. To this purpose, we re-write the usual system using (1.8), so that we deal with

$$-\Delta u + [|\nabla\theta - q\mathbf{A}|^2 + m^2 - (\omega - q\phi)^2] u + N'(u) = 0, \quad (5.95)$$

$$-\Delta\phi = q(\omega - q\phi)u^2, \quad (5.96)$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(l\nabla\theta - q\mathbf{A})u^2. \quad (5.97)$$

*Proof of Theorem 1.7.* If  $\mathbf{A} = \mathbf{0}$ , in [11] a variational identity for solutions of (5.95) was given. However, the same identity holds when  $\mathbf{A} \neq \mathbf{0}$ , and it reads as follows:

$$\begin{aligned} 0 = & - \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla\phi|^2 dx - 3\Omega \int_{\mathbb{R}^3} u^2 dx \\ & - 3q \int_{\mathbb{R}^3} (2\omega - q\phi)\phi u^2 dx + 6 \int_{\mathbb{R}^3} F(u) dx, \end{aligned} \quad (5.98)$$

where we have set  $\Omega = m^2 - \omega^2$ ,  $F(s) = \int_0^s f(t) dt$  and

$$f(u) = -|l\nabla\theta - q\mathbf{A}|^2 u - N'(u).$$

Since  $\phi$  solves (5.96), we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 dx = q \int_{\mathbb{R}^3} (\omega - q\phi)u^2 \phi dx; \quad (5.99)$$

substituting (5.99) in (5.98) and computing  $F(u)$  we get

$$\begin{aligned} 0 = & - \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} [3\Omega + 5q\omega\phi - 2q^2\phi^2 + 3|l\nabla\theta - q\mathbf{A}|^2] u^2 dx \\ & - 6 \int_{\mathbb{R}^3} N(u) dx. \end{aligned} \quad (5.100)$$

By (2.52) and (2.53), if  $N \geq 0$  and  $\omega^2 < m^2$ , we get  $u \equiv 0$ .

Moreover, since  $u$  solves (5.95), we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}|^2 u^2 dx + m^2 \int_{\mathbb{R}^3} u^2 dx - \int_{\mathbb{R}^3} (\omega - q\phi)^2 u^2 + \int_{\mathbb{R}^3} N'(u)u dx = 0; \quad (5.101)$$

substituting the expression  $\int_{\mathbb{R}^3} |\nabla u|^2 dx$  taken from (5.101) into (5.100), we obtain

$$\begin{aligned} 0 = & q \int_{\mathbb{R}^3} (q\phi - 3\omega)u^2 \phi dx - 2 \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}|^2 u^2 dx \\ & + 2(\omega^2 - m^2) \int_{\mathbb{R}^3} u^2 dx + \int_{\mathbb{R}^3} [N'(u)u - 6N(u)] dx. \end{aligned} \quad (5.102)$$

Thanks to (2.52) and (2.53), all the terms in (5.102) are non-positive if  $\omega^2 < m^2$ ,  $N'(s)s - 6N(s) \leq 0$ ; hence  $u \equiv 0$ .

Finally, when  $N'(s)s \geq 2N(s)$ , we proceed as follows: from (5.101) we get

$$\begin{aligned} \Omega \int_{\mathbb{R}^3} u^2 dx = & - \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |l\theta - q\mathbf{A}|^2 u^2 dx \\ & - 2q\omega \int_{\mathbb{R}^3} u^2 \phi dx + q^2 \int_{\mathbb{R}^3} u^2 \phi^2 dx - \int_{\mathbb{R}^3} N'(u)u dx. \end{aligned} \quad (5.103)$$

Substituting (5.103) into (5.100) we get

$$0 = 2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} qu^2 \phi(\omega - q\phi) dx + \int_{\mathbb{R}^3} [3N'(u)u - 6N(u)] dx. \quad (5.104)$$

Analogously, now all the coefficients are non-negative, and thus  $u \equiv 0$ .  $\square$

## References

- [1] A. Azzolini and A. Pomponio, *Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations*, Topol. Methods Nonlinear Anal. (2010)
- [2] V. Benci and D. Fortunato, *Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations*, Rend. Accad. Naz. Lincei, Mat. Appl. **20**, 243–279 (2009)
- [3] V. Benci and D. Fortunato, *Solitary waves in Abelian Gauge Theories*, Adv. Nonlinear Stud. **8** (2008), no. 2, 327–352.
- [4] V. Benci and D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations*, Rev. Math. Phys. **14** (2002), 409–420.
- [5] V. Benci and D. Fortunato, *Spinning  $Q$ -Balls for the Klein-Gordon-Maxwell Equations*, Commun. Math. Phys. **295**, 639–668 (2010)

- [6] V. Benci and D. Fortunato, *Three dimensional vortices in abelian gauge theories*, Nonlinear Analysis **70**, 4402–4421 (2009).
- [7] V. Benci and D. Fortunato, *Towards a Unified Field Theory for Classical Electrodynamics*, Arch. Rational Mech. Anal. **173** (2004), 379–414.
- [8] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I - Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), no. 4 , 313–345.
- [9] T. Cazenave and P.L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), no. 4, 549–561.
- [10] S. Coleman, *Q-Balls*. Nucl. Phys. B262, 263–283 (1985); erratum: B269, 744–745 (1986)
- [11] T. D’Aprile T. and D. Mugnai, *Non-existence results for the coupled Klein–Gordon–Maxwell equations*, Adv. Nonlinear Stud. **4** (2004), no. 3, 307–322.
- [12] T. D’Aprile and D. Mugnai, *Solitary Waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations*, Proc. R. Soc. Edinb. Sect. A **134** (2004), 1–14.
- [13] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon and H.C. Morris, *Solitons and Non-linear Wave Equations*, Academic Press, London, New York, 1982.
- [14] I. Ekeland, *On the variational principle*. J. Math. Anal. Appl. **47**, 324–353 (1974).
- [15] M. Esteban, V. Georgiev and E. Sere, *Stationary waves of the Maxwell–Dirac and the Klein–Gordon–Dirac equations*, Calc. Var. Partial Differential Equations **4** (1996), 265–281.
- [16] M. Esteban and P.L. Lions, *A compactness lemma*, Nonlinear Anal. **7**, 381–385 (1983).
- [17] B. Felsager, *Geometry, particles and fields*, Odense University Press 1981.
- [18] S. Klainerman and M. Machedon, *On the Maxwell–Klein–Gordon equation with finite energy*, Duke Math. J. **74** (1994), no. 1, 19–44.
- [19] E.H. Lieb and H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), no. 1, 147–174.
- [20] P.L. Lions, *The concentration–compactness principle in the calculus of variations. The locally compact case. Part I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145.
- [21] P.L. Lions, *The concentration–compactness principle in the calculus of variations. The locally compact case. Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no.4, 223–283.



- [22] E. Long, *Existence and stability of solitary waves in non-linear Klein-Gordon-Maxwell equations*, Rev. Math. Phys. **18** (2006), 747–779.
- [23] M. Machedon and J. Sterbenz, *Almost optimal local well-posedness for the  $(3+1)$ -dimensional Maxwell-Klein-Gordon equations*, J. Amer. Math. Soc. **17** (2004), no. 2, 297–359.
- [24] D. Mugnai, *Coupled Klein-Gordon and Born-Infeld type equations: looking for solitary waves*, R. Soc. Lond. Proc. Ser. A **460** (2004), 1519–1528.
- [25] D. Mugnai, *The pseudorelativistic Hartree equation with a general nonlinearity: existence, non existence and variational identities*, 28 p, submitted.
- [26] D. Mugnai, *Solitary waves in Abelian Gauge Theories with strongly nonlinear potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 1055–1071.
- [27] R. Rajaraman, *Solitons and instantons*, North Holland, Amsterdam, Oxford, New York, Tokyo, 1988.
- [28] I. Rodnianski and T. Tao, *Global regularity for the Maxwell-Klein-Gordon equation with small critical Sobolev norm in high dimensions*, Comm. Math. Phys. **251** (2004), no. 2, 377–426.
- [29] G. Rosen, *Particlelike solutions to nonlinear complex scalar field theories with positive-definite energy densities*, J. Math. Phys. **9** (1968), 996–998.
- [30] S. Selberg, *Almost optimal local well-posedness of the Maxwell-Klein-Gordon equations in  $1+4$  dimensions*, Comm. in Partial Differential Equations **27** (2002), no. 5-6, 1183–1227.
- [31] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), no. 2, 149–162.
- [32] C.H. Taubes, *On the Yang-Mills-Higgs equations*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), no. 2, 295–297.